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EFFECTIVE LOWER BOUNDS FOR SOME LINEAR FORMS

BY

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ABSTRACT. It is proved that if $1, \alpha, \beta$ are numbers, linearly independent over the rationals, in a real cubic number field, then given any real number $d \geq 2$, for any integers x_0, x_1, x_2 such that $|\text{norm}(x_0 + \alpha x_1 + \beta x_2)| \leq d$, there exist effectively computable numbers $c > 0$ and $k > 0$ depending only on α and β such that $|x_1 x_2|(\log|x_1 x_2|)^{k \log d}|x_0 + \alpha x_1 + \beta x_2| > c$ holds whenever $x_1 x_2 \neq 0$. It would be of much interest to remove the dependence on d in the exponent of $\log|x_1 x_2|$, for then, among other things, one could deduce, for cubic irrationals, a stronger and effective form of Roth's Theorem.

1. **Introduction.** Let $\|x\|$ denote the nonnegative distance from x to the nearest integer. The well-known theorem of Roth [11] asserts that for any real irrational algebraic number α and any $\epsilon > 0$, there is a constant $c > 0$ such that $q^{1+\epsilon}\|q\alpha\| > c$ holds for all integers $q > 0$. Schmidt [12] has generalized this result to any number of dimensions. He proved that if $\alpha_1, \ldots, \alpha_n$ are any real algebraic numbers such that $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over the rationals, then for any $\epsilon > 0$ there is a constant $c_n > 0$ such that

$$|q_1 q_2 \cdots q_n|^{1+\epsilon}\|q_1\alpha_1 + \cdots + q_n\alpha_n\| > c_n \tag{1}$$

holds for all nonzero integers $q_1, \ldots, q_n$. He also proved a dual result, namely that, under the same hypotheses on $\alpha_1, \ldots, \alpha_n$, for any $\epsilon > 0$ there is a constant $c_n' > 0$ such that

$$q^{1+\epsilon}\|q\alpha_1\| \cdots \|q\alpha_n\| > c_n' \tag{2}$$

holds for all integers $q > 0$.

The theorems of Roth and Schmidt are *noneffective*, that is, the constants c, c_n, c_n' cannot be effectively computed. Much recent work has been done on the problem of establishing effective lower bounds for $\|q_1\alpha_1 + \cdots + q_n\alpha_n\|$, where $\alpha_1, \ldots, \alpha_n$ are algebraic numbers. All of the effective lower bounds obtained so far are appreciably weaker than (1) (see the survey articles by Baker [1], [3]).

The main result of this paper is:

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THEOREM 1. *Suppose* 1, α *and* β *are numbers, linearly independent over the rationals, in a real cubic number field. Given any number* $d \geqslant 2$, *for any integers* x_0, x_1, x_2 *such that* $|\mathrm{norm}(x_0 + \alpha x_1 + \beta x_2)| \leqslant d$, *there exist effectively computable numbers* $c > 0$ *and* $k > 0$ *depending only on* α *and* β *such that*

$$|x_1 x_2|(\log|x_1 x_2|)^{k \log d}|x_0 + \alpha x_1 + \beta x_2| > c \tag{3}$$

holds whenever $|x_1 x_2| > 1$.

The hypotheses of Theorem 1 are very restrictive, but inequality (3) is both effective and *stronger* than (1). Indeed, (3) is nearly best possible, for it follows from some work of Peck [10] that if $1, \alpha, \beta$ are numbers, linearly independent over the rationals, in a real cubic number field, then the inequality

$$|x_1 x_2|(\log|x_1 x_2|)^{1/2}|x_0 + \alpha x_1 + \beta x_2| < 1$$

has infinitely many solutions in integers x_0, x_1, x_2 with $x_1 x_2 \neq 0$. (Peck actually proved that $q \log q \|q\alpha\| \, \|q\beta\| < 1$ has infinitely many solutions with $q > 0$; one obtains the dual inequality stated above by using a transference principle of Mahler [9]. The argument, in a slightly different setting, is given in Cassels and Swinnerton-Dyer [6, p. 82].) Thus for fixed d inequality (3) can only be improved by decreasing the exponent of $\log|x_1 x_2|$.

Of course it is natural to hope that the factor $\log d$ in the exponent of $\log|x_1 x_2|$ could be removed altogether, for then Theorem 1 would hold with no condition on the norm of $x_0 + \alpha x_1 + \beta x_2$. This hope is plausible from a metrical point of view, because it is known (see Gallagher [8] and Spencer [13]) that for almost all pairs of real numbers α, β an inequality of the form (3) holds with $k \log d$ replaced by an absolute constant. If Theorem 1 were strengthened in this way, then it would follow from an application of Mahler's transference principle that there exist effectively computable numbers $a > 0$ and $c > 0$ depending only on α and β such that $q(\log q)^a \|q\alpha\| \, \|q\beta\| > c$ holds for all $q > 0$. As an immediate corollary, we would have an inequality for all cubic irrationals which is both effective and stronger than Roth's Theorem.

2. **Preliminaries.** Let F denote the real cubic field containing α and β. For any number γ in F, we let $\gamma^{(0)} = \gamma$, $\gamma^{(1)} = \gamma'$, $\gamma^{(2)} = \gamma''$ denote the conjugates of γ. Let G denote the smallest field containing $\alpha, \alpha', \alpha'', \beta, \beta'$ and β''. If F is a cyclic (totally real) cubic field, then $F = G$; otherwise G is a normal field of degree 6, which is totally real if and only if F is.

From here until §5 we suppose that F is *totally real.* The case of non-totally real F (§5) turns out to be simpler.

Let M denote the module $\{x_0 + \alpha x_1 + \beta x_2 : x_0, x_1, x_2 \text{ rational integers}\}$ and let $T = \{\gamma : \gamma \text{ in } F, M \text{ contains } \gamma M\}$ denote the coefficient ring of M. It follows from Dirichlet's unit theorem [4, p. 112, Theorem 5] that there exist two units θ_1 and θ_2 in T such that every unit in T has the form $\pm\theta_1^m\theta_2^n$, where m and n are rational integers.

Let

$$\mathbf{R} = \det\begin{bmatrix}\log|\theta_1'| & \log|\theta_2'| \\ \log|\theta_1''| & \log|\theta_2''|\end{bmatrix}. \tag{4}$$

The regulator of T is $|\mathbf{R}|$, so $|\mathbf{R}|$ is not zero and is independent of the choice of the fundamental units θ_1, θ_2 for T [4, p. 115].

If θ_1, θ_2 is a pair of fundamental units for T, then so is $\theta_1^p\theta_2^q, \theta_1^r\theta_2^s$ for integers p, q, r, s such that $ps - qr = \pm 1$. Hence for any $\epsilon_0 > 0$ (however small) and any $L > 0$ (however large) it is possible to choose θ_1 in the pair θ_1, θ_2 of fundamental units so that

$$||\theta_1/\theta_1''| - 1| < \epsilon_0 \quad \text{and} \quad |\theta_1'| > L. \tag{5}$$

Later on in the paper (Lemmas 5 and 6) we shall require the inequalities (5) for certain explicitly calculable ϵ_0 and L. We assume from now on that a fixed pair θ_1, θ_2 of fundamental units for T has been chosen in such a way that (5) holds for the relevant ϵ_0 and L. It is convenient to assume further that θ_1, θ_2 have been chosen in such a way that θ_1 and θ_2 both have norm +1 and

$$\mathbf{R} = \text{regulator of } T > 0 \tag{6}$$

(of course given any pair θ_1, θ_2 we can satisfy (6) by simply replacing θ_2 by θ_2^{-1}, if necessary).

Define

$$D(1) = -\log|\theta_2'| + \log|\theta_2''|, \qquad D(2) = \log|\theta_1'| - \log|\theta_1''|.$$

Neither $D(1)$ nor $D(2)$ is zero. (Of course, $D(2) > 0$ by (5) if L is large enough; we assume this is the case.) Indeed, more generally, if $\gamma \neq \pm 1$ is any element of F, then $\gamma\gamma'^2 \neq \pm 1$. For $\gamma\gamma'^2 = \pm 1$ implies $\gamma'\gamma''^2 = \gamma''\gamma^2 = \pm 1$, and solving these equations gives $\gamma^9 = \pm 1$, which implies $\gamma = \pm 1$. We define η_0, η_1, η_2 (numbers which will be encountered later on) as follows:

$$\eta_i = |\theta_1^{(i)}|^{D(1)}|\theta_2^{(i)}|^{D(2)} \qquad (0 \leqslant i \leqslant 2). \tag{7}$$

It is easy to see that

$$\log\eta_0 = -2\mathbf{R}, \qquad \log\eta_1 = \log\eta_2 = \mathbf{R}. \tag{8}$$

For any pair of rational integers m, n define

$$R(m, n) = \theta_1^m\theta_2^n.$$

Since 1 is in M, the numbers $R(m, n)$ are in M for all m, n and we have

$$R(m, n) = a_0(m, n) + a_1(m, n)\alpha + a_2(m, n)\beta \tag{9}$$

for some rational integers $a_i(m, n)$ $(0 \leqslant i \leqslant 2)$.

We now need to consider some results about the solutions in integers x_0, x_1, x_2 of the inequality

$$|x_0 + \alpha x_1 + \beta x_2| \max(x_1^2, x_2^2) < Q, \tag{10}$$

where Q is any constant $\geqslant 1$. Later on we will use these results in dealing with the expression $|x_0 + \alpha x_1 + \beta x_2|\, |x_1 x_2|$ in the proof of Theorem 1.

We first recall the well-known result [5, p. 79] that for $1, \alpha, \beta$ linearly independent in a real cubic field there is a constant κ satisfying $0 < \kappa < 1$ and depending only on α and β such that

$$\kappa \leqslant |x_0 + \alpha x_1 + \beta x_2| \max(x_1^2, x_2^2) \tag{11}$$

holds for all integers x_0, x_1, x_2 with x_1 and x_2 not both zero.

Next we need the following lemma.

Lemma 1. *Let a be any nonzero rational number. Then in the module M there is a finite (possibly empty) set of numbers $\mu_1, \ldots, \mu_k$ with norm a such that every solution γ of* norm $\gamma = a$, *γ in M, has a unique representation in the form $\gamma = \mu_i R(m, n)$ for some i $(1 \leqslant i \leqslant k)$ and some m, n.*

Proof. See [4, p. 118, Theorem 1].

From now on we use $c_1, c_2, \ldots$ to denote positive constants which depend at most on α, β, θ_1 and θ_2. Of course, since we have assumed that θ_1 and θ_2 have been chosen, we could as well say that $c_1, c_2, \ldots$ depend at most on α and β. However, we shall sometimes wish to know that certain constants are the same *whatever* choice of θ_1 and θ_2 might have been made; this is not in general true for $c_1, c_2, \ldots$.

If we let $\xi = x_0 + \alpha x_1 + \beta x_2$ and $\xi^{(i)} = x_0 + \alpha^{(i)} x_1 + \beta^{(i)} x_2$ $(0 \leqslant i \leqslant 2)$, then clearly

$$\max_{0 \leqslant i \leqslant 2} |\xi^{(i)}| \leqslant c_1 \max_{0 \leqslant i \leqslant 2} |x_i|. \tag{12}$$

Since in Theorem 1 we are concerned with a lower bound for $|\xi|$, we may clearly assume that x_0 is always taken to be the nearest integer to $\alpha x_1 + \beta x_2$. Thus $|\xi| \leqslant \frac{1}{2}$ holds and so

$$\max_{0 \leqslant i \leqslant 2} |x_i| \leqslant c_2 \max(|x_1|, |x_2|). \tag{13}$$

It follows from (12) and (13) that if (10) holds, then

$$|\xi| \max(|\xi'|^2, |\xi''|^2) < Q(c_1 c_2)^2. \tag{14}$$

By (14), norm ξ is bounded if ξ satisfies (10), so by Lemma 1 in looking at (10) we need only consider solutions of

$$|\delta R(m, n)| \max(b_1(m, n)^2, b_2(m, n)^2) < Q \tag{15}$$

where

$$\xi = \delta R(m, n) = b_0(m, n) + b_1(m, n)\alpha + b_2(m, n)\beta \tag{16}$$

for some rational integers $b_i(m, n)$ $(0 \leqslant i \leqslant 2)$ and where δ runs through some finite set (depending on Q) in M. Call this set $\Delta(Q)$.

We can be more explicit about the membership of $\Delta(Q)$. Let

$$\tau = \min\{|\text{norm } \gamma| : \gamma \neq 0 \text{ in } M\}. \tag{17}$$

Of course the set on the right-hand side is discrete, and $\tau > 0$. It now follows from (14) that if $|\xi| \leqslant \frac{1}{2}$ and (10) holds, then

$$\tau \leqslant |\text{norm } \delta| < Q(c_1 c_2)^2. \tag{18}$$

Thus by Lemma 1, for each of the finite number of values d of norm δ satisfying (18), $\Delta(Q)$ contains some finite number of elements $\mu_1^{[d]}, \ldots, \mu_{k(d)}^{[d]}$, all of norm d.

We use the notation (a_{ij}) for a matrix with entry a_{ij} in the ith row and jth column.

LEMMA 2. *The integers $a_i(m, n)$ $(0 \leqslant i \leqslant 2)$ in* (9) *satisfy*

$$a_i(m, n) = \sum_{j=0}^{2} a_{ij} R^{(j)}(m, n) \qquad (0 \leqslant i \leqslant 2) \tag{19}$$

where the coefficient matrix $A = (a_{ij})$ $(0 \leqslant i, j \leqslant 2)$ satisfies

$$A = \begin{bmatrix} 1 - \alpha a_{10} - \beta a_{20} & 1 - \alpha' a_{11} - \beta' a_{21} & 1 - \alpha'' a_{12} - \beta'' a_{22} \\ D_M^{-1}(\beta' - \beta'') & D_M^{-1}(\beta'' - \beta) & D_M^{-1}(\beta - \beta') \\ D_M^{-1}(\alpha'' - \alpha') & D_M^{-1}(\alpha - \alpha'') & D_M^{-1}(\alpha' - \alpha) \end{bmatrix}; \tag{20}$$

here

$$D_M = \det \begin{bmatrix} \alpha' - \alpha & \beta' - \beta \\ \alpha'' - \alpha & \beta'' - \beta \end{bmatrix} \tag{21}$$

is one of the square roots of the discriminant of the module M.

The integers $b_i(m, n)$ $(0 \leqslant i \leqslant 2)$ in (16) *satisfy*

$$b_i(m, n) = \sum_{j=0}^{2} b_{ij} R^{(j)}(m, n) \qquad (0 \leqslant i \leqslant 2) \tag{22}$$

where the coefficient matrix $B = (b_{ij})$ $(0 \leqslant i, j \leqslant 2)$ is given by

$$B = (\delta^{(j)} a_{ij}) \qquad (0 \leqslant i, j \leqslant 2). \tag{23}$$

PROOF. Let C denote the matrix on the right-hand side of (21). We have the identity

$$C\begin{bmatrix} a_1(m, n) \\ \\ a_2(m, n) \end{bmatrix} = \begin{bmatrix} R'(m, n) - R(m, n) \\ \\ R''(m, n) - R(m, n) \end{bmatrix}.$$

Multiplying both sides by the inverse matrix C^{-1} gives (19) and (20) (the case $i = 0$ of (19) follows at once from the cases $i = 1, 2$ and (9)).

To prove (22) and (23) we apply the argument above to the identity

$$C\begin{bmatrix} b_1(m, n) \\ \\ b_2(m, n) \end{bmatrix} = \begin{bmatrix} \delta' R'(m, n) - \delta R(m, n) \\ \\ \delta'' R''(m, n) - \delta R(m, n) \end{bmatrix}.$$

We note that any ratio of two numbers in the same row or column of A or B is in the field G. If F is cyclic (so $F = G$), then D_M is a rational integer and $a_{i1} = a'_{i0}, a_{i2} = a''_{i0}$ $(0 \leqslant i \leqslant 2)$. In this case all the entries in A and B are themselves in F.

3. **A technical lemma.** It is clear from Lemma 1 that in $\Delta(Q)$ defined above we may replace any of the elements $\mu_i^{[d]}$ of norm d by $R(m, n)\mu_i^{[d]}$, where $R(m, n)$ is any unit in the coefficient ring T. Let $\Delta(Q, d)$ denote the subset of $\Delta(Q)$ made up of those numbers with norm $\leqslant d$. It follows from (18) that $\Delta(Q, d)$ has a *fixed* number of elements for all $Q > d/(c_1c_2)^2$. Thus we may suppose $\Delta(Q, d)$ is independent of Q for Q large enough; call this set $\Delta^*(d)$. In the proof of Theorem 1 it will be necessary to ensure that for any μ in $\Delta^*(d)$, the absolute value of the ratio of any two conjugates of μ in F is bounded by a constant independent of the choice of θ_1, θ_2. This can be achieved by replacing each element μ of $\Delta^*(d)$ by a suitable $R(m, n)\mu$; the following lemma is needed to show that such a replacement is always possible.

LEMMA 3. *Let μ be any element of M. Then there exists a number ζ in M such that ζ/μ is a unit in T and*

$$\max_{0 \leqslant i,j \leqslant 2} |\zeta^{(i)}/\zeta^{(j)}| \leqslant c_3 \tag{24}$$

where c_3 is a constant depending only on the module M.

PROOF. Let φ_1, φ_2 be any pair of fundamental units for T. If we define $\zeta = \varphi_1^x \varphi_2^y \mu$, then we can choose x and y so that

$$|\zeta^{(i)}| \leqslant c\, |\text{norm}\, \zeta|^{1/3} \qquad (0 \leqslant i \leqslant 2),$$

where c is a constant depending only on M [4, pp. 122–123]. Then ζ/μ is a unit in T and clearly (24) holds.

Given any number μ in $\Delta^*(d)$, we can replace μ by the corresponding number ζ given in Lemma 3. Thus we can assume

$$\text{for any } \mu \text{ in } \Delta^*(d), \quad \max_{0 \leqslant i,j \leqslant 2} |\mu^{(i)}/\mu^{(j)}| \leqslant c_3, \tag{25}$$

where c_3 depends only on α and β, and not on the choice of θ_1, θ_2 in T. Furthermore, if μ_1 is in $\Delta^*(d_1)$ and, for some $d_2 \geqslant d_1$, μ_2 is a number in $\Delta^*(d_2)$ such that $\mu_2/\mu_1 = (p/q)R(m, n)$ for some unit $R(m, n)$ in T, where p, q are rational integers and $p/q \geqslant 1$, then we can assume that $\mu_2/\mu_1 = p/q$. We shall say that each number μ in $\Delta^*(d)$ is *canonical* if this last assumption and (25) both hold.

We suppose from now on that *every element of* $\Delta^*(d)$ *is canonical.*

4. **Proof of Theorem 1, totally real case.** The first major ingredient in the proof of Theorem 1 is the following theorem of Baker [2, I].

THEOREM 2 (BAKER). *Let $\sigma_1, \ldots, \sigma_n$ be nonzero algebraic numbers with degrees at most Σ and let the heights of $\sigma_1, \ldots, \sigma_{n-1}$ and σ_n be at most A' and A $(\geqslant 2)$, respectively. Then for some effectively computable number $C > 0$ depending only on n, Σ and A', the inequalities*

$$0 < |p_1 \log \sigma_1 + \cdots + p_n \log \sigma_n| < C^{-\log A \log P}$$

have no solution in rational integers $p_1, \ldots, p_n$ with absolute value at most P $(\geqslant 2)$.

We have stated the theorem in the general form given by Baker, though we require only the special case $n = 3$, $p_3 = -1$ for the following lemma. We do not need the further refinement of Theorem 2 in the case $p_n = -1$ which is given by Baker [2, II, Theorem 2].

LEMMA 4. *Given any number $d \geqslant 2$, if δ is canonical, $x_1x_2 \neq 0$ and $\xi = \delta R(m, n) = x_0 + \alpha x_1 + \beta x_2$ satisfies $|\text{norm}\ \xi| \leqslant d$, then*

$$|x_1x_2(x_0 + \alpha x_1 + \beta x_2)| \geqslant c_4/(\max(|m|, |n|))^{c_5 \log d}. \tag{26}$$

PROOF. We may assume that $|x_1| \geqslant |x_2|$; an argument parallel to what follows takes care of the case $|x_1| < |x_2|$. Since

$$|x_1x_2\xi| = |x_2/x_1|\, |x_1^2|\, |\xi| = |x_2/x_1|\, |\xi| \max(x_1^2, x_2^2)$$

it follows from (11) that the lemma is true if we can prove

$$|x_2/x_1| > c_6/(\max(|m|, |n|))^{c_5 \log d}. \tag{27}$$

To do this, we first observe that by (16) and Lemma 2

$$x_1 = a_{10}\xi + a_{11}\xi' + a_{12}\xi'', \quad x_2 = a_{20}\xi + a_{21}\xi' + a_{22}\xi''. \tag{28}$$

We may assume that $|\xi|$ is very small, for otherwise (26) is trivial. If $|\xi|$ is small enough, it follows that

$$|x_2| \geqslant \tfrac{1}{2}|a_{21}\xi' + a_{22}\xi''| \quad \text{and} \quad |x_1| \leqslant c_7 \max(|\xi'|, |\xi''|),$$

so

(29) $$|x_2/x_1| \geqslant c_8 |a_{21}\xi' + a_{22}\xi''|/\max(|\xi'|, |\xi''|).$$

We may assume $|\xi''| \geqslant |\xi'|$; a similar argument takes care of the case $|\xi''| < |\xi'|$. Thus we need a lower bound for the absolute value of

$$a_{21}(\xi'/\xi'') + a_{22} = a_{22}((b_{21}\,\theta_1'^m\theta_2'^n/b_{22}\theta_1''^m\theta_2''^n) + 1).$$

Applying Theorem 2 with $n = 3$, $\sigma_1 = |\theta_1'/\theta_1''|$, $\sigma_2 = |\theta_2'/\theta_2''|$, $\sigma_3 = |b_{22}/b_{21}|$, $p_3 = -1$, we find that for any integers m, n not both zero either

(30) $$|b_{21}\theta_1'^m\theta_2'^n| = |b_{22}\theta_1''^m\theta_2''^n|$$

or

(31) $$|m\log|\theta_1'/\theta_1''| + n\log|\theta_2'/\theta_2''| - \log|b_{22}/b_{21}|| \geqslant 1/(\max(|m|, |n|))^\nu,$$

where $\nu > 0$ is a constant depending on the degree and height of b_{22}/b_{21}. In fact, Theorem 2 shows that ν is of the form $k_1\log(\text{height}(b_{22}/b_{21}))$ where k_1 depends on the degree but not the height of b_{22}/b_{21}. However, $b_{22}/b_{21} = \delta''(\alpha' - \alpha)/\delta'(\alpha - \alpha'')$ is a number in G. Since δ is canonical, it follows from (25) with $\mu = \delta$ that the heights of δ' and δ'' are less than $k_2 d$, where k_2 depends only on α and β. This implies (after some easy estimates) that the height of b_{22}/b_{21} is less than $k_3 d^\Gamma$, where k_3 depends only on α and β, and Γ is an absolute constant. Hence in (31) we can take $\nu = c_9 \log d$. Now (27) follows easily after exponentiating (31) and applying the inequality $e^x > 1 + x$.

It remains to consider the case when (30) holds, so Theorem 2 does not apply. If (30) is true with the absolute value signs removed, then $a_{21}(\xi'/\xi'') + a_{22} = 2a_{22}$, so (27) is trivially true. If, on the other hand, $a_{21}\xi' + a_{22}\xi'' = 0$, then for some integers r and s we have (using (20))

(32) $$-\frac{b_{22}}{b_{21}} = \frac{\delta''(\alpha - \alpha')}{\delta'(\alpha - \alpha'')} = \left(\frac{\theta_1'}{\theta_1''}\right)^r \left(\frac{\theta_2'}{\theta_2''}\right)^s.$$

Suppose (32) holds for two different canonical values of δ, say δ_1 and δ_2 (with possibly different values of r and s, of course). It follows that for some integers t and u we have

$$(\delta_1''/\delta_2'')\theta_1''^t\theta_2''^u = (\delta_1'/\delta_2')\theta_1'^t\,\theta_2'^u\,.$$

The two sides of this equality are numbers in different conjugate fields of F; hence both numbers are equal to some rational number q/p. This means $\delta_1(p/q)\theta_1^t\theta_2^u = \delta_2$, but since δ_1 and δ_2 are canonical we must have $t = u = 0$ (see §3). Therefore even if (32) holds for various values of δ, the integers r and s do not change. It follows that

$$b_{21}\theta_1'^m\theta_2'^n + b_{22}\theta_1''^m\theta_2''^n = 0$$

is possible for at most *one* integer pair m, n. Even if there is such an exceptional pair m, n, we can still ensure that (27) holds by adjusting the constant c_6, if necessary. This completes the proof of the lemma.

In view of Lemma 4, if we could prove that

$$\log|x_1 x_2| > c \max(|m|, |n|) \tag{33}$$

holds for some constant c depending only on α and β, then Theorem 1 would be proved. An inequality of the form (33) follows easily from (28) and the fact that $\xi = \delta\theta_1^m\theta_2^n$. However, the constant c obtained in this way depends on δ (and so on d); with such a constant, inequality (33) is useless for proving Theorem 1. In order to get around this difficulty, we need more information about m and n. The following theorem, which is the second major ingredient in the proof of Theorem 1, gives us what we need.

THEOREM 3. *Suppose there exists* $\xi = x_0 + \alpha x_1 + \beta x_2$ *with norm of absolute value* w *such that* (15) *and* (16) *hold, with* δ *canonical. Then* $|m - \lambda n| \leqslant K$ *where* $\lambda = D(1)/D(2)$ *and* K *is a constant depending only on* Q, w, α, β, *and* θ_1. *We can take*

$$K = \frac{\log(hQ/w)}{D(2)} = \frac{\log(hQ/w)}{\log|\theta_1'/\theta_1''|}$$

where h *depends only on* α *and* β, *and not on the choice of* θ_1 *and* θ_2.

PROOF. This is essentially a special case of the theorem of Cusick [7, p. 19], except for the evaluation of K. I repeat the proof of [7] with the changes needed to get an explicit K.

The norm $\xi\xi'\xi''$ has absolute value w, so by (14)

$$|\xi| \min_{i=1,2} |\xi^{(i)}|^2 = w^2 \left(|\xi| \max_{i=1,2} |\xi^{(i)}|^2\right)^{-1} > (w/c_1c_2)^2 Q^{-1}. \tag{34}$$

Combining (14) and (34) gives

$$\max_{i=1,2} |\xi^{(i)}| < (c_1c_2)^2 Q w^{-1} \min_{i=1,2} |\xi^{(i)}| \tag{35}$$

for any $\xi = \delta R(m, n)$ which satisfies (15).

If (15) holds then (25) with $\mu = \delta$ and (35) give

$$\max_{i=1,2} |R^{(i)}(m, n)| < (c_1c_2)^2 c_3 Q w^{-1} \min_{i=1,2} |R^{(i)}(m, n)|;$$

it follows that

$$|\log|R'(m, n)| - \log|R''(m, n)|| < \log((c_1c_2)^2 c_3 Q/w),$$

and so

$$\begin{aligned} m\log|\theta_1'| + n\log|\theta_2'| &= \log|R''(m, n)| + \epsilon, \\ m\log|\theta_1''| + n\log|\theta_2''| &= \log|R''(m, n)|, \end{aligned} \tag{36}$$

where $|\epsilon| < \log((c_1c_2)^2 c_3 Q/w)$.

The system (36) has for its determinant the regulator $\mathbf{R}$ given by (4). Thus solving for m and n gives

$$m = D(1)\mathbf{R}^{-1}\log|R''(m, n)| + \epsilon\mathbf{R}^{-1}\log|\theta_2''|,$$
$$n = D(2)\mathbf{R}^{-1}\log|R''(m, n)| - \epsilon\mathbf{R}^{-1}\log|\theta_1''|.$$

Hence

$$|m - \lambda n| = |\epsilon\mathbf{R}^{-1}(\log\eta_2)D(2)^{-1}| \leqslant D(2)^{-1}\log((c_1c_2)^2c_3Q/w)$$

where η_2 is defined by (7), and for the inequality we have used (8) and the upper bound on $|\epsilon|$. It is evident that c_1, c_2 and c_3 depend only on α and β, so we have Theorem 3 with $h = (c_1c_2)^2c_3$.

As was remarked earlier, we obviously need to consider only those solutions of (10) where x_0 is the nearest integer to $\alpha x_1 + \beta x_2$. Assuming this, we divide the integer solutions to (10) into classes according to the size of the left-hand side. In the first class the left-hand side of (10) is $\geqslant \kappa$ (defined in (11)) and < 1; in the second class the left side of (10) is $\geqslant 1$ and < 2; and in the Nth class ($N = 2, 3, \ldots$) the left side of (10) is $\geqslant 2^{N-2}$ and $< 2^{N-1}$. Thus any integer pair $(x_1, x_2) \neq (0, 0)$ is in one of the classes, since it will satisfy (10) for Q large enough. Using this division into classes and Theorem 3, we prove the following lemma, which gives inequality (33) with a suitable constant when $n > 0$.

LEMMA 5. *If* $\xi = \delta R(m, n) = x_0 + \alpha x_1 + \beta x_2$ *with* δ *canonical,* $n > 0$ *and* $x_1x_2 \neq 0$, *then* $\log|x_1x_2| > c_{10}n$.

PROOF. As in the proof of Lemma 4, we may assume $|x_1| \geqslant |x_2|$. Suppose x_1, x_2 is in the Nth class of solutions to (10), so (10) holds for some $Q < 2^{N-1}$; also suppose $|\text{norm}\,\delta| = w$. By Theorem 3 and (7), (8)

$$|R(m, n)| = \exp(-2\mathbf{R}D(2)^{-1}n)\,|\theta_1|^{\epsilon}, \qquad |\epsilon| \leqslant K. \tag{37}$$

Since $\theta_1\theta_1'\theta_1'' = 1$ and, *if* ϵ_0 *in* (5) *is chosen small enough and* L *in* (5) *is chosen large enough,* $|\theta_1''| \approx |\theta_1| < 1$, we have $2\log|\theta_1| + \log|\theta_1'| \approx 0$ and (using the definition of K in the first inequality, the definition $D(2) = \log|\theta_1'/\theta_1''|$ in the second inequality, and $Q < 2^{N-1}$ in the third inequality)

$$\begin{aligned} |\theta_1|^{\epsilon} &\leqslant \exp(-\log(hQ/w)D(2)^{-1}\log|\theta_1|) \\ &\leqslant c_{11}\exp(-|\log|\theta_1|/\log|\theta_1\theta_1'^2||\log Q) \\ &\leqslant c_{12}\exp(-N(1/3 + \epsilon_1)\log 2), \end{aligned} \tag{38}$$

where $\epsilon_1 > 0$ is a small number depending on ϵ_0 in (5) (ϵ_1 decreases if ϵ_0 is chosen smaller).

By (18), $|\delta\delta'\delta''| < Q(c_1c_2)^2$, so it follows from (25) and the fact that δ is canonical that

$$|\delta| \leqslant c_{13}Q^{1/3} \leqslant c_{14}\exp((N/3)\log 2). \tag{39}$$

Since x_1, x_2 is in the Nth class of solutions, we have

(40) $$2^{N-2} \leqslant |\delta R(m, n)| x_1^2$$

if $N > 1$ (for $N = 1$, the left side of the above inequality should be the number κ defined in (11)). Applying (37), (38) and (39) to (40), we obtain

$$\begin{aligned} x_1^2 &\geqslant c_{15} 2^N \exp(-N(2/3 + \epsilon_1)\log 2 + 2RD(2)^{-1}n) \\ &> c_{16} \exp(2RD(2)^{-1}n), \end{aligned}$$

where the second inequality holds if ϵ_1 is small enough. The lemma follows at once.

The inequality of Theorem 1 follows from Lemmas 4 and 5 *provided* $n > 0$. If $n \leqslant 0$, then a result stronger than Theorem 1 is true, as we shall show using the following lemma.

LEMMA 6. *Suppose* $\xi = \delta R(m, n) = x_0 + \alpha x_1 + \beta x_2$ *satisfies* (10) *with* δ *canonical and* x_1, x_2 *is in the jth class of solutions*, $j \leqslant N$. *If* $n \leqslant N$ *and* $|x_1| \geqslant |x_2| > 0$, *then*

$$|x_2/x_1| > c_{17}/2^N.$$

PROOF. Since x_1, x_2 is in the Nth class of solutions, we have

(41) $$|\delta R(m, n)| x_1^2 < 2^{N-1}.$$

Since δ is canonical and by (18) has norm of absolute value at least τ, (25) implies

(42) $$|\delta| \geqslant c_{18} |\text{norm}\, \delta|^{1/3} \geqslant c_{19}.$$

Equality (37) holds as before, and by the same reasoning used to prove (38) (the only difference is that here we want a lower bound) we obtain

(43) $$|\theta_1|^\epsilon \geqslant c_{20} \exp(-N(1/3 + \epsilon_1)\log 2).$$

Since $D(2) = \log|\theta_1'/\theta_1''|$ is large *if* L *in* (5) *is chosen large*, we have

$$\exp(-2RD(2)^{-1}n) \geqslant \exp(-2RD(2)^{-1}N) \geqslant \exp(-\epsilon_2 N \log 2)$$

where $\epsilon_2 > 0$ is a small number depending on L in (5) (ϵ_2 decreases if $D(2)^{-1}$ is chosen smaller, that is, if L is chosen larger). Using this inequality, (37), (42) and (43), we see that (41) gives

$$x_1^2 < c_{21} \exp(((4/3) + \epsilon_1 + \epsilon_2)N \log 2).$$

If ϵ_1 and ϵ_2 are small enough and $x_1 x_2 \neq 0$, this easily gives the inequality of the lemma.

Now suppose $n \leqslant 0$ and the hypotheses of Lemma 6 hold (of course there is no loss of generality in assuming the hypothesis $|x_1| \geqslant |x_2|$). Since x_1, x_2 is in the Nth class of solutions, we have

$$|x_1 x_2 \xi| = |x_2/x_1|\,|x_1^2|\,|\xi| \geqslant c_{22}|x_2/x_1|\,2^N.$$

It follows from this and Lemma 6 that the inequality of Theorem 1 holds even without the logarithm factor, when $n \leqslant 0$.

It is, of course, not surprising that this stronger result holds in the cases $n \leqslant 0$ because of (37) and the fact that we need only consider those pairs m, n for which $|\delta R(m, n)| \leqslant ½$. Since $RD(2)^{-1} > 0$, this means that for a given δ only a finite number of $n \leqslant 0$ come into consideration.

5. **Proof of Theorem 1, nontotally real case.** In this section we suppose that F is nontotally real. The proof of Theorem 1 in this case follows the general pattern of the proof for the totally real case (§§2 to 4), but the details are much simpler. Therefore we only sketch the proof.

The coefficient ring T of the module M has only one fundamental unit, say θ, which we suppose is < 1. Lemmas 1 and 2 are valid with the obvious changes (for example, $R(m, n)$ must be replaced by θ^m). Lemma 3 holds as before, and the definition of canonical is the same.

We do not need any analogue of the conditions (5) or of Theorem 3 in the proof. We obtain an analogue of Lemma 4 by assuming the hypotheses of that lemma and using Theorem 2 with $n = 2$, $\sigma_1 = \theta'/\theta''$, $\sigma_2 = b_{22}/b_{21}$ and $p_2 = -1$. We obtain

$$|x_1 x_2(x_0 + \alpha x_1 + \beta x_2)| \geqslant c_{23}/|m|^{c_{24}\log d}.$$

We divide the solutions of (10) into classes as before; then analogues of Lemmas 5 and 6 are easily obtained. This completes the proof of the nontotally real case of Theorem 1.

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